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On the stability of gas bubbles oscillating non-spherically in a compressible liquid

H. J. RATH

Institut für Mechanik, Universität Hannover, Appelstraße 11, D-3000 Hannover 1, Germany

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SUMMARY
This paper describes the non-spherical free and forced oscillations of a gas bubble in a compressible liquid. Generally two different cases of oscillations are possible: spherically radial motion and surface oscillations. The deviation from spherical shape is assumed to be small and is given by a spherical harmonic. Included in the theoretical model are the effect of surface tension, the compressibility of the liquid and the gas. Stability and threshold conditions for the shape oscillations are given.

1. Introduction

The fundamental nature of vibratory cavitation and underwater explosions is the transient growth and collapse of individual gas bubbles in liquids. Recent investigations [1-7, 28, 29, 30] have shown that the influence of the fluid compressibility can be very strong for the free [27] and forced radial oscillations of a gas bubble in a liquid. The results of the compressible theory are in very good agreement with the experimental data [3, 4, 7, 8]. If a linearized analysis of the spherical oscillations is carried out, the effect of sound radiation is found to be very modest except at high frequency. The large effect of compressibility appears only in the non-linear regime. On physical reasoning one expects the effect of compressibility on the non-spherical oscillations to be quite modest, because this motion would contribute multipole radiation, and not monopole radiation as in the spherical case and any large effect would only appear in a non-linear analysis, similar to the spherical case. Bubbles at rest or undergoing small-amplitude pulsations tend to have a spherical shape, because of surface tension. The spherical shape of a bubble is unstable, however, if the amplitude of the pulsation is sufficiently large. It is well known that shape instability occurs when the sound-pressure amplitude exceeds a threshold value that depends on the bubble radius and the acoustic frequency. The problem of the stability of a plane interface between two incompressible fluids of different densities in accelerated motion has been solved by Taylor [9]. The corresponding problem for a nearly spherical interface has been discussed by Binnie [10], Plesset [11], Strasberg [12, 13], Birkhoff [15], Eller and Crum [16] and Hsieh et al. [17]. Numerical results of the Plesset equations are given by Strube [18]. For a slightly viscous incompressible liquid the non-spherical bubble oscillations have been calculated by Prosperetti [19, 20], Francescutto et al. [21] and Ceschia [22]. Excellent photographs of surface waves are given by Kornfeld and Suvorov [23] and Hullin [24]. In all these studies of non-spherical bubble dynamics the compressibility of the liquid is neglected. Several investigations have shown that for stronger radial bubble oscillations the mean damping parameter is the fluid compressibility and not the viscosity effect, so that in this paper the compressibility of the liquid is taken into account for the mathematical model describing non-spherical bubble oscillations. Hsieh [31] has already discussed the effect of compressibility of the liquid on the non-spherical motion of gas bubbles.

For the following investigations thermal and diffusion effects and the viscosity of the liquid are neglected. We consider a sphere of gas of initial radius a_0 surrounded by an unbounded fluid initially at rest. First we calculate the velocity potential for the strictly spherical radial motion. Then we can add the potential for the distortion of the interface and we obtain the velocity potential for the whole system.

2. Case of a strictly spherical interface

The equations for conservation of mass and momentum in spherical coordinates are given by [25]

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial r} + \frac{2}{r}u = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \tag{2}$$

where ρ is the density of the liquid, r is the radius, u the velocity of the fluid particles in the r-direction, p is the pressure in the liquid and t is the time. First we calculate the velocity potential ϕ for the strictly spherical interface of the bubble. Hence, as long as liquid velocities are small compared to the velocity of sound in the liquid, we obtain from Eq. (1) and (2) the wave equation [7, 25]

$$-c^{-2}\phi_{tt} + \phi_{rr} + \frac{2}{r}\phi_r = 0 \tag{3}$$

in which ϕ is the velocity potential and c is the constant sound speed of the liquid. It is very interesting that Taylor [14] has shown that for a flow produced by a uniformly expanding sphere, the agreement between the flow based on the wave equation and exact flow was excellent even for very large velocities of the sphere.

For an incompressible fluid ($c \to \infty$) we obtain from (3) the Laplace equation. For the pressure difference inside and outside nearby the bubble surface we obtain

$$p_g - p_a = \frac{2\sigma}{a} \tag{4}$$

where p_g is the gas pressure inside of the bubble, p_a is the pressure in the liquid nearby the bubble surface, σ is the surface tension and a is the momentary radius of the spherical cavity. It is assumed that the gas pressure inside the bubble is uniformly distributed. It should be noted that (4) is only valid for a strictly spherical interface (see Sec. 3). The gas inside the bubble is compressed according to a polytropic gas law. Furthermore the mass of gas is assumed to be constant. Then with (4) for the pressure in the liquid on the bubble surface we write [2, 6, 8]

$$p_a = p_{g0}\left(\frac{a}{a_0}\right)^{-3\kappa} - \frac{2\sigma}{a} + p_A \sin(\omega t) \tag{5}$$

in which p_{g0}, a_0, κ, p_A and ω denote respectively initial gas pressure, radius of the bubble at rest ($t = 0$), polytropic exponent, sound pressure amplitude and the angular frequency of the sound field. For the free bubble oscillation the last term on the right hand side of (5) is zero. The initial gas pressure p_{g0} is calculated from (4) for $t = 0$ and $a = a_0$,

$$p_{g0} = p_0 + \frac{2\sigma}{a_0}\ . \tag{6}$$

A particular solution of (3) for diverging waves in the r-direction is

$$\phi = \frac{1}{r}\, F(t - (r - a_0)/c) \tag{7}$$

where F is a function of the retarted $(t - (r - a_0)/c)$. With (7) and (3), Keller and Kolodner [7] have derived the following nonlinear second-order ordinary differential equation for the case of a strictly spherical interface of the bubble:

$$\begin{aligned} a\ddot{a}\left(1 - \frac{\dot{a}}{c}\right) + \frac{3}{2}\dot{a}^2\left(1 - \frac{1}{3}\frac{\dot{a}}{c}\right) - \left(1 + \frac{\dot{a}}{c}\right)\left(\frac{p_a - p_0}{\rho_0}\right) \\ - \frac{a}{c}\left(\frac{p_a - p_0}{\rho_0}\right)^{\bullet} = 0. \end{aligned} \tag{8}$$

A dot denotes a derivative with respect to time, ρ_0 is the density of the liquid, p_0 is its initial pressure and $\dot{a}$ is the radial velocity of the bubble surface. Now we can insert the pressure p_a (see (5)) into (8) and we obtain the differential equation

$$\begin{aligned} a\ddot{a}\left(1 - \frac{\dot{a}}{c}\right) + \frac{3}{2}\dot{a}^2\left(1 - \frac{1}{3}\frac{\dot{a}}{c}\right) - \frac{p_{g0}}{\rho_0}\left(\frac{a}{a_0}\right)^{-3\kappa}\left(1 + (1 - 3\kappa)\frac{\dot{a}}{c}\right) + \frac{1}{\rho_0}\frac{2\sigma}{a} \\ + \left(1 + \frac{\dot{a}}{c}\right)\frac{p_0}{\rho_0} - \frac{p_A}{\rho_0}\left[\left(1 + \frac{\dot{a}}{c}\right)\sin(\omega t) + \frac{a\omega}{c}\cos(\omega t)\right] = 0. \end{aligned} \tag{9}$$

A very similar equation is also obtained by Lastman and Wentzel [28] for a tension wave $p_s f(t) = p_s \exp(-t/t_0)$. They obtained the following result [28]:

$$a\ddot{a}(\dot{a}-c)+\frac{3}{2}\dot{a}^2\left(\frac{1}{3}\dot{a}-c\right)+\frac{k}{\rho_0}\left(\frac{4\pi}{3}a^3\right)^{-\kappa}[(1-3\kappa)\dot{a}+c]$$

$$-\frac{c}{\rho_0}\frac{2\sigma}{a}-(c+\dot{a})\frac{p_0}{\rho_0}+(\dot{a}+c)\frac{p_s f(t)}{\rho_0}+\frac{a p_s}{\rho_0}\frac{df(t)}{dt}=0 \tag{10}$$

with $k=(p_0+2\sigma/a_0)\cdot[(4\pi/3)a_0^3]^\kappa$.

A comparison of (10) with (9) shows that for the present paper $p_s f(t)=p_A\sin(\omega t)$. For an incompressible fluid ($c\to\infty$) we obtain from (9) the well-known incompressible RPNNP-model [2, 6]

$$a\ddot{a}+\frac{3}{2}\dot{a}^2=\frac{p_{g0}}{\rho_0}\left(\frac{a}{a_0}\right)^{-3\kappa}-\frac{2\sigma}{\rho_0 a}-\frac{p_0}{\rho_0}+\frac{p_A}{\rho_0}\sin(\omega t)=\frac{p_a-p_0}{\rho_0}. \tag{11}$$

It is very interesting that if we multiply eq. (11) with $\dot{a}/c$ and subtract from (9) we obtain Herring's equation for the first order in $\dot{a}/c$ [6, 5, 35]:

$$a\ddot{a}\left(1-2\frac{\dot{a}}{c}\right)+\frac{3}{2}\dot{a}^2\left(1-\frac{4}{3}\dot{a}\right)+\frac{2\sigma}{\rho_0 a}\left(1-\frac{\dot{a}}{c}\right)-\frac{p_{g0}}{\rho_0}\left(\frac{a}{a_0}\right)^{-3\kappa}\left[1-3\kappa\frac{\dot{a}}{c}\right]$$
$$+\frac{p_0}{\rho_0}-\frac{p_A}{\rho_0}\left[\sin(\omega t)+\frac{a\omega}{c}\cos(\omega t)\right]=0. \tag{12}$$

The initial conditions for (10) are given by

$$a(0)=a_{max}\ ;\ \dot{a}(0)=0 \tag{13a}$$

for the free oscillation and

$$a(0)=a_0\ ;\ \dot{a}(0)=0 \tag{13b}$$

for the forced oscillation. Once $a(t)$ is known, F and F' can be calculated [7] as

$$F=-a^2\dot{a}+\frac{a^2}{c}\left[\frac{\dot{a}^2}{2}+\frac{1}{\rho_0}(p_a-p_0)\right], \tag{14}$$

$$F'=-a\left[\frac{\dot{a}^2}{2}+\frac{1}{\rho_0}(p_a-p_0)\right]. \tag{15}$$

Then the velocity potential is given by

$$\phi=\frac{F(t-(r-a_0)/c)}{r}=-\frac{a^2\dot{a}}{r}+\frac{a^2}{c-r}\left[\frac{\dot{a}^2}{2}+\frac{1}{\rho_0}(p_a-p_0)\right]. \tag{16}$$

For $c \to \infty$ we obtain from (14) the velocity potential for an incompressible liquid which has been used by several investigators.

3. Case of a non-spherical interface

The origin of the coordinate system is taken at the center of the spherical interface. When the bubble is distorted from the spherical shape of instantaneous radius a to a surface with radius vector of magnitude r_s we have

$$r_s = a + b\ Y_n \tag{17}$$

where $a(t)$ is the spherical interface radius vector, $b(t)$ is the amplitude vector of the surface disturbance and Y_n is the spherical harmonic of order n. It is assumed that $|b(t)| \ll a(t)$. The stability analysis given here will be limited to the first order in b. Then the fluid velocity at the interface in the radial direction is given by

$$u = \dot{r}_s = \dot{a} + \dot{b}\ Y_n, \tag{18}$$

because the difference between the normal component of the liquid velocity at the interface and the radial velocity u is of second order in b, so that we can neglect this difference. Now we introduce a potential which corresponds to a disturbance which decreases away from the interface in the outward direction,

$$\phi_s = \phi + \phi_n \qquad \text{for } r > r_s, \tag{19}$$

where ϕ_s is the potential for the whole problem, ϕ is the potential for the strictly spherical interface and ϕ_n is the potential of the surface disturbance. We have

$$\phi_s = \frac{F(t - (r - a_0)/c)}{r} + \frac{A\,(t - (r - a_0)/c)}{r^{n+1}} Y_n \qquad \text{for } r > r_s. \tag{20}$$

F is known from (14) and the quantity A is determined by the following boundary conditions:

$$\left(\frac{\partial \phi_s}{\partial r}\right)_{r_s} = \dot{a} + \dot{b}\ Y_n = \dot{r}_s \qquad \text{at } r = r_s, \tag{21}$$

$$\frac{p_a - p_0}{\rho_0} = -\left(\frac{\partial \phi_s}{\partial t}\right)_{r_s} - \frac{1}{2}\left(\frac{\partial \phi_s}{\partial r}\right)_{r_s}^2 \qquad \text{at } r = r_s. \tag{22}$$

It may be noted that p_a is not identical with (5). For a perturbed sphere the sum of the curvatures up to the first-order correction may be written as [25]

$$\frac{1}{a_1} + \frac{1}{a_2} = \frac{2}{a} + \frac{(n-1)(n+1)}{a^2}\, b\ Y_n \tag{23}$$

where a_1 and a_2 are principal radii of curvature of the bubble surface. Instead of (5) we can write

$$p_a = p_{g0}\left(\frac{a}{a_0}\right)^{-3\kappa} - \frac{2\sigma}{a} - \frac{(n-1)(n+1)\sigma}{a^2}\, b\, Y_n + p_A \sin(\omega t). \tag{24}$$

Substituting (20) into (21) we obtain

$$\left(\frac{\partial \phi_s}{\partial r}\right)_{r_s} = -\frac{1}{r_s^2} F - \frac{1}{r_s c} F' - \frac{n+1}{r_s^{n+2}} A\, Y_n - \frac{1}{c r_s^{n+1}} A'\, Y_n = \dot{a} + \dot{b}\, Y_n \tag{25}$$

in which the prime denotes a derivative with respect to the retarded time. Then from (22) we obtain

$$\frac{p_a - p_0}{\rho_0} = -\frac{1}{r_s} F' - \frac{1}{r_s^{n+1}} A'\, Y_n - \frac{1}{2}\dot{r}_s^2. \tag{26}$$

The functions A and A' can be determined from (25) and (26), which yield

$$A = \frac{a^{n+2}}{n+1}\left\{2\left[-\dot{a} + 1/c\left(\frac{1}{2}\dot{a}^2 + \frac{1}{\rho_0}(p_a - p_0)\right)\right]\frac{b}{a} - \dot{a}(1 - \dot{a}/c)\frac{\dot{b}}{\dot{a}}\right\}, \tag{27}$$

$$A' = a^{n+1}\left\{-\left(\frac{1}{2}\dot{a}^2 + \frac{1}{\rho_0}(p_a - p_0)\right)\frac{b}{a} - \dot{a}^2\frac{\dot{b}}{\dot{a}}\right\}. \tag{28}$$

It may be noted that it is possible with the help of (20), (27) and (28) to calculate the pressure distribution in the liquid for $r > r_s$. Now we determine the equation of motion for the bubble boundary. Substituting (14) and (27) into (22) and with (15), (21) and (28) we obtain Eq. (9) to zero order in b. For the first order in b we obtain

$$\begin{aligned}
&\left\{-\frac{\ddot{a}}{a}(n-1)\left(1 - \frac{\dot{a}}{c}\right) + \frac{n-1}{a c \rho_0}\left[-3\kappa\, p_{g0}\left(\frac{a}{a_0}\right)^{-3\kappa}\frac{\dot{a}}{a} + p_A\, \omega\cos(\omega t)\right] + \right.\\
&\left. + \frac{(n+1)(n-1)(n+2)}{\rho_0 a^3}\,\sigma\left(1 - \frac{\dot{a}}{c}\,\frac{n(n+3)}{(n+1)(n+2)}\right)\right\} b + \left\{3\,\frac{\dot{a}}{a}\left(1 - \frac{1}{6}(n+5)\frac{\dot{a}}{c}\right) - \right.\\
&- \frac{\ddot{a}}{c} + \frac{n-1}{a c \rho_0}\left(p_{g0}\left(\frac{a}{a_0}\right)^{-3\kappa} + p_A \sin(\omega t) - p_0\right) + \\
&\left. + \frac{(n-1)((n^2-1)+(n+3))}{\rho_0 a^2}\,\sigma\,\frac{1}{c}\right\}\dot{b} + \left(1 - \frac{\dot{a}}{c}\right)\ddot{b} = 0.
\end{aligned} \tag{29}$$

This differential equation determines the stability of a small-amplitude distortion of a spherical interface. For an incompressible liquid ($c \to \infty$) we obtain from (9) the incompressible RPNNP-model [2], [6] for a strictly spherical shape (see (12)). For an incompressible fluid (29) becomes

$$\ddot{b} + 3\,\frac{\dot{a}}{a}\,\dot{b} + (n-1)\left[\frac{(n+1)(n+2)}{\rho_0 a^3}\,\sigma - \frac{\ddot{a}}{a}\right] b = 0, \tag{30}$$

a result already obtained by Plesset [11] and Birkhoff [15].

4. Stability of the non-spherical interface

Equation (9) shows that, to first order in b, the shape distortion does not affect the radius. The equation of motion for the surface disturbance (29) is a second-order differential equation with periodic coefficients. The natural frequency of the strictly radial oscillations of the bubble in an incompressible liquid is given by Minnaert [26]

$$\omega_0 = \frac{1}{a_0}\sqrt{\frac{1}{\rho_0}\left[3\kappa\left(p_0 + \frac{2\sigma}{a_0}\right) - \frac{2\sigma}{a_0}\right]}, \tag{31}$$

and for the surface modes [25]

$$\omega_n = \sqrt{\frac{(n-1)(n+1)(n+2)\sigma}{\rho_0\, a_0^3}}\,. \tag{32}$$

It may be noted that (31) and (32) are results obtained by means of the linear theory and (32) is only valid for a = constant, that is, pure surface waves. In absence of an ultrasonic field ($\omega = 0$) for a compressible fluid (29) becomes

$$\begin{aligned} &\left\{-\frac{\ddot{a}}{a}(n-1) - \frac{(n-1)\,3\kappa\, p_{g0}(a/a_0)^{-3\kappa}}{c\,a^2\rho_0(1-\dot{a}/c)}\,\dot{a} + \frac{(n+1)(n-1)(n+2)}{\rho_0\, a^3(1-\dot{a}/c)}\,\sigma\right.\\ &\left.\times\left[1 - \frac{\dot{a}}{c}\,\frac{n(n+3)}{(n+1)(n+2)}\right]\right\} b + \left\{3\,\frac{\dot{a}}{c}\,\frac{(1 - 1/6\,(n+5)\,\dot{a}/c)}{(1-\dot{a}/c)} - \frac{\ddot{a}}{c(1-\dot{a}/c)}\right.\\ &\left.+ \frac{(n-1)(p_{g0}(a/a_0)^{-3\kappa} - p_0)}{c\,a\,\rho_0(1-\dot{a}/c)} + \frac{(n-1)((n^2-1)+(n+3))}{c(1-\dot{a}/c)\rho_0 a^2}\,\sigma\right\}\dot{b} + \ddot{b} = 0. \end{aligned} \tag{33}$$

For a bubble of constant radius a_0, eq. (33) reduces to the equation for a damped harmonic oscillator. The natural frequency is given by (32) and the damping constant is

$$d = \frac{(n-1)}{2\,c\,a_0^2\,\rho_0}\,(n^2 + n + 4)\,\sigma. \tag{34}$$

For the first order in $\dot{a}/c$ equation (29) becomes

$$\begin{aligned}
&\ddot{b} + \left\{ 3\frac{\dot{a}}{a}\left(1 - \frac{1}{6}(n-1)\frac{\dot{a}}{c}\right) - \frac{\ddot{a}}{\dot{a}}\frac{\dot{a}}{c} + \frac{(n-1)\dot{a}}{\dot{a}\,a\,\rho_0\,c}\left(p_{g0}\left(\frac{a}{a_0}\right)^{-3\kappa}\right.\right.\\
&\left.\left. + p_A\sin(\omega t) - p_0\right) + \frac{(n-1)((n^2-1)+(n+3))}{\rho_0 a^2 \dot{a}}\frac{\dot{a}}{c}\sigma\right\}\dot{b} + \left\{-\frac{\ddot{a}}{a}(n-1)\right.\\
&+ \frac{(n-1)}{a^2\rho_0}\frac{\dot{a}}{c}\left(-3\kappa p_{g0}\left(\frac{a}{a_0}\right)^{-3\kappa} + \frac{p_A\,\omega a}{\dot{a}}\cos(\omega t)\right)\\
&\left. + \frac{(n+1)(n-1)(n+2)}{\rho_0 a^3}\sigma\left(1 + \frac{\dot{a}}{c}\frac{2}{(n+1)(n+2)}\right)\right\} b = 0.
\end{aligned} \tag{35}$$

Now we introduce the new variable g defined as

$$b = g\exp\left(-1/2\int_{t_0}^{t} H(t^*)\,dt^*\right) \tag{36}$$

where $H(t)$ is the coefficient of $\dot{b}$ in (35). Then with (9) we obtain (35) in the more suitable form

$$\ddot{g} - G(t)\;g = 0 \tag{37}$$

where $G(t)$ is defined as

$$\begin{aligned}
G(t) &= (n+1/2)\frac{\ddot{a}}{a}\left(1 - \frac{n-1}{2n+1}\frac{\dot{a}}{c}\right) + 3/4\frac{\dot{a}^2}{a^2}\left(1 - 2/3\,(n-1)\frac{\dot{a}}{c}\right)\\
&- \frac{(n+1)(n-1)(n+2)}{\rho_0 a^3}\sigma\left(1 + \frac{2-(n-1)^2(n+2)}{2(n+1)(n-1)(n+2)}\frac{\dot{a}}{c}\right)\\
&- 1/2\frac{n-2}{a^2\rho_0}\frac{\dot{a}}{c}\left(-3\kappa p_{g0}\left(\frac{a}{a_0}\right)^{-3\kappa} + \frac{p_A\,\omega a}{\dot{a}}\cos(\omega t)\right)\\
&+ \frac{n-1}{a^2\rho_0}\frac{\dot{a}}{c}\left(p_{g0}\left(\frac{a}{a_0}\right)^{-3\kappa} + p_A\sin(\omega t)\right).
\end{aligned} \tag{38}$$

For an incompressible liquid (38) becomes

$$G(t) = \left(n + \frac{1}{2}\right)\frac{\ddot{a}}{a} + \frac{3\dot{a}^2}{4a^2} - \frac{(n+1)(n-1)(n+2)}{\rho_0 a^3}\sigma \tag{39}$$

which is the well-known function $G(t)$ given by Plesset [11] and Eller et al. [16]. Using a mean polytropic exponent $\kappa = 4/3$, Eq. (38) becomes for the free oscillation ($\omega = 0$):

$$G(t) = (n + \frac{1}{2}) \frac{\ddot{a}}{a} \left(1 - \frac{n-1}{2n+1} \frac{\dot{a}}{c}\right) + \frac{3\dot{a}^2}{4a^2} \left(1 - \frac{2}{3}(n-1)\frac{\dot{a}}{c}\right)$$

$$- \frac{(n+1)(n-1)(n+2)}{a^3 \rho_0} \sigma \left(1 + \frac{2-(n-1)^2(n+2)}{2(n+1)(n-1)(n+2)} \frac{\dot{a}}{c}\right) \tag{40}$$

$$- \frac{n-3}{a^2 \rho_0} \frac{\dot{a}}{c} p_{g0} \left(\frac{a}{a_0}\right)^{-4} - \frac{n-1}{a^2 \rho_0} p_0 \frac{\dot{a}}{c} .$$

Equation (37) is Hill's equation [36] which is characteristic of parametric resonances. By means of Floquet's theory [36] the general solution of (37) may be written as

$$g(t) = C_1 e^{\mu t} \alpha(t) + C_2 e^{-\mu t} \beta(t) \tag{41}$$

in which $\alpha(t)$ and $\beta(t)$ are periodic functions of t and μ is the characteristic exponent. The periodic solution is asymptotically stable if the real parts of $-\sigma \pm \mu$ are negative, where 2σ is the constant term in $H(t)$ (Eq. (35)). The solution of (37) is unstable if one of the real parts of $-\sigma \pm \mu$ is positive. When p_A/p_0 is small compared with unity, the strictly radial oscillation will be approximately simply harmonic. A linearized calculation of Eq. (9) gives with (29) the Mathieu equation for the first order in p_A/p_0. This equation, given in [33], is characteristic of parametric resonances. The results show an influence of the fluid compressibility on the stability of the surface waves [33]. Furthermore a comparison of the results with Hsieh's incompressible threshold values [32] is given in [33]. In [34] it is shown that the theory with compressibility is in very good agreement with the experimental data of Hullin [24].

In the present paper we have used the wave equation and as a result we obtain two independent differential equations for the bubble oscillations. A quantitative comparison of (35) with Eq. (26) of Hsieh [31] is not possible because the wave nature of the solution ϕ_n is suppressed by Hsieh's approximation. Neglecting the wave nature of ϕ_n then from Hsieh's Eq. (26) we obtain with

$$p = p_0 - \rho_0 \left[- \frac{(a^2 \dot{a})^{\cdot}}{r} + \frac{(a^2 \dot{a})^2}{2 r^4} \right] , \tag{42}$$

given by Keller and Kolodner [7], the 'incompressible' result given by Eq. (30), but the functions a, $\dot{a}$, $\ddot{a}$ are different in our theory. These functions are given by the solution of (9) in the compressible theory.

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